$$F=+\sqrt{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2}=+\sqrt{X^2+Y^2+Z^2},$$

acts in space so that the direction cosines are

$$\cos\alpha=\frac{-\frac{\partial V}{\partial x}}{F},\quad \cos\beta=\frac{-\frac{\partial V}{\partial y}}{F},\quad \cos\gamma=\frac{-\frac{\partial V}{\partial z}}{F}.$$

Thus, if the potential function for a system of particles is known the attraction is completely defined and the components of the attraction in any given direction may be computed at once.

§ 2. Laplace's Equation for Two Variables.

Solutions of Laplace's equation are called *harmonic functions.* Among the harmonic functions are those which depend upon two variables only. If $V(x, y)$ is a potential function the corresponding distribution of matter must be such that the z-component of attraction vanishes at every point. This may be realized in three ways:

1). Consider a thin uniform rod of finite length perpendicular to the x-y plane and bisected by it. At every point in the x-y plane the z-component of attraction is zero and the potential function will be a function of x and y alone. From the definition of potential it is clear that the number of rods of the kind described may be increased indefinitely and the number of variables in the potential function will not be changed.

2). If the rods are conceived to extend indefinitely in both directions parallel to the z-axis the potential function depends upon x and y only for all points in space since the z-component of attraction is zero for every point in space.

3). Suppose that the z-component of attraction due to a material system becomes neutralized by an equal opposing force at every point in the x-y plane. These forces become lost forces in the sense used by D'Alembert in his "Traité de Dynamique." The effective forces must then act along curves which lie in the x-y plane. The potential function for the effective forces in the x-y plane is therefore a harmonic function of the two variables x and y. For example, consider the x-y plane as a face of a finite smooth slab of metal lying in a horizontal position. If at time t_0 water stands on the plane, the depth varying from point to point, the downward pressure of the water at each point is counteracted by the opposing upward pressure of the plane. The difference of pressure of water at the x-y plane will cause motions of the water particles over the x-y plane from points of higher to points of lower pressure. If this instantaneous condition can be perpetuated by locating sources and sinks of appropriate strength at proper

positions in the x-y plane so-called *steady-streaming* will occur. If the metallic plate is thin and if it conducts electricity the positive and negative electrodes of any number of cells may be stationed at arbitrary points on the plane. Steady streaming of electricity will begin and continue as long as the electrodes are not moved and the relative potentials of the electrodes are not changed. The form of the potential function for two variable will be considered next.

§ 3. Logarithmic Potential.

Let $t(x, y)$ be a function which may or may not be harmonic. Let it be assumed that, if t is not harmonic, there is a function of t that is harmonic. The necessary condition which t must fulfill may be found as follows:

Differentiating $f(t)$,

$$\frac{\partial f(t)}{\partial x} = f'(t) \cdot \frac{\partial t}{\partial x}$$

$$\frac{\partial^2 f(t)}{\partial x^2} = f''(t) \left(\frac{\partial t}{\partial x}\right)^2 + f'(t) \frac{\partial^2 t}{\partial x^2}.$$

Likewise, $$\frac{\partial^2 f(t)}{\partial y^2} = f''(t) \left(\frac{\partial t}{\partial y}\right)^2 + f'(t) \frac{\partial^2 t}{\partial y^2}.$$

Adding and using the fact that $f(t)$ is harmonic,

$$\triangle f(t) = \left[\left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial y}\right)^2\right] f''(t) + \left[\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2}\right] f'(t) = 0.$$

Therefore, $$\frac{\dfrac{\partial^2 t}{\partial x^2} + \dfrac{\partial^2 t}{\partial x^2}}{\left(\dfrac{\partial t}{\partial x}\right)^2 + \left(\dfrac{\partial t}{\partial x}\right)^2} = -\frac{f''(t)}{f'(t)}.$$

For brevity let L denote the differential operator in the left member of their relation. The right hand member is a function of t alone. It is evident then that the necessary condition which t must satisfy in order that there may be found a harmonic function of t is that $L\,t$ shall be a function of t alone, say $Q(t)$. When t satisfies this condition the required harmonic function $f(t)$ may be found by integration in the form

$$f(t) = k_1 \int e^{-\int Q dt} d\,t + k_2.$$

Suppose, for example, that $t = x^3 + y^3$. Then

$$Lt = \frac{2\,(x+y)}{3\,(x^4+y^4)}.$$

But this is not a function of the argument t alone, consequently there is no harmonic function of x^3+y^3.

Again, consider $t = (x^2+y^2)^{\frac{1}{2}} = r.$

Then $Lt = \frac{1}{t} = Q(t).$

Hence, $f(t) = k_1 \int e^{-\int dt/t} d\,t + k_2 = k_1 \log t + k_2.$

Remembering that the potential function involves mass and distance, it is clear that so long as the mass elements of the system remain constant and their relative positions remain fixed, the potential function must vary only with r. But the form of the potential function of r only has been shown to be

$$V = k_1 \log r + k_2,$$

the constants k_1 and k_2 depending upon the mass and potential units respectively. From its form the potential function of two variables is called *Logarithmic Potential.*

§ 4. Conjugate Functions.

The equation $V(x, y) = c_1$ defines for each value of c_1 a curve in the x-y plane. The set of curves $U(x\ y) = c_2$ which are orthogonal to the first set must satisfy the equation

$$\frac{\partial V}{\partial x} \cdot \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \cdot \frac{\partial U}{\partial y} = 0,$$

which is the necessary and sufficient condition for orthogonal functions.*

*Scheffer's "Auwendung der Differential und Integral Rechuung auf Geometrie," Vol. I, p. 110.

Cauchy and Riemann found as the necessary and sufficient condition that $W=V+iU$ should be an analytic function of the complex variable $z=x+iy$ the following differential equations:

$$\frac{\partial V}{\partial x}=\frac{\partial U}{\partial y} \text{ and } \frac{\partial V}{\partial y}=-\frac{\partial U}{\partial x}.$$

From these equations the condition for orthogonality follows at once. $U(x\,y)$ is said to be the conjugate function of the function $V(x,\,y)$. It is seen that if U is conjugate of V, then $-V$ will be conjugate of U, but in either case U and V are orthogonal functions. If U is orthogonal to V for every point within a given region of the $x\,y$ plane it is possible to choose the sign of U so that

$$V+i\,U=W$$

shall be an analytic function of z within the same region of the $x\,y$ plane.

The attraction due to a system of particles has already been given for the general case. For such distributions as lead to potential functions of two variables only, the magnitude of the attraction is given by the expression

$$F=\sqrt{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2},$$

and its direction referred to the positive x-axis is given for each point by the expression

$$\tan\theta=\frac{\dfrac{\partial V}{\partial y}}{\dfrac{\partial V}{\partial x}}.$$

The slope of the tangent to the equipotential curves, $V=c_1$, is given by the expression

$$\tan\theta_1=-\frac{\dfrac{\partial V}{\partial x}}{\dfrac{\partial V}{\partial y}}.$$

Hence, $$|\theta_1-\theta|=\frac{\pi}{2}.$$

Thus the force acts along the normal to the equipotential curve at each point. For each distribution of matter there correspond two sets of functions which are mutually orthogonal and which if taken with proper signs are conjugate, one of the other. These functions are the potential function and the function which defines the stream lines. Conjugate functions are of fundamental importance in the theory of logarithmic potential as well as in the theory of analytic functions. Laplace's equation is the necessary condition in both fields. Riemann founded his theory of functions on this equation, and both he and Klein* drew many of their theorems on analytic functions from the laws and phenomena of conservative forces and steady currents. Holzmüller† has written several articles on various phases of this Physico-Function-theoretic unity. In at least two of these articles he deals with families of Cassinian ovals of higher order and with their orthogonal curves which he calls hyperbolas of higher order. It makes these higher plane curves much more interesting to find that they have an important physical significance.

§5. Analytic Functions and Steady Streaming in a Plane.

As already shown the potential function for a thin rod is given by

$$v=k_1 \log r+k_2,$$

or, briefly, $$v=k_1 \log r,$$

where k_1 depends upon the density of the rod and the universal gravitational constant. The conjugate function is

$$U=k_1\theta=k_1 \arctan\frac{y}{x}.$$

Then $$V+i\,U=k_1 \log r+i\,k_1\theta=k_1 \log z.$$

So the simplest case of logarithmic potential leads to a simple logarithmic function on the function-theoretic side. Let the units be so chosen that

*Klein: On Riemann's Theory of Algebraic Functions.

†Isogonal verwandtschaften; Ueber einen Satz der Functionentheorie und seine Anwendung auf isothermische Kurven-systeme und auf einige Theorie der Mathematische Physik. Zeitschrift für Math. und Physik, Vol. 42, p. 217. Ueber die Abbildung und die lemniscatischen Coordinaten nter Ordnug. Crelle, 83, p. 38; Zusammenhang der Hyperbeln und Lemniscaten hoeheren Ordunng mit dem Ausgangpunkte der Funktionen Theorie. Zeitschrift für Math. und Physik, Vol. 29, p. 120.

$k_1=1$ if density equals one,
$k_1=2$ if density equals two, etc.

Then if k_1 is the density of a rod whose axis passes through the $x\,y$ plane at the point $(x_1\,y_1)$ the potential function is

$$V=k_1 \log r_1=k_1 \log \sqrt{(x-x_1)^2+(y-y_1)^2},$$

and the stream function is

$$U=k_1\,\theta_1=k_1 \text{ arc tan } \frac{y-y_1}{x-x_1}.$$

The corresponding analytic function is

$$W=k_1 \log (z-a), \quad a=x_1+i\,y_1.$$

Potential is a scalar quantity and hence the potential due to a set of n rods with densities $k_1, k_2 \ldots k_i \ldots k_n$, respectively, and passing through the x-y plane at the points $a_1 \equiv (x_1\ y_1), \ldots a_i \equiv (x_i\ y_i), \ldots a_n \equiv (x_n\ y_n)$ is given by

$$V=\sum_{i=1}^{n} k_i \log r_i=\log \prod_{i=1}^{n} r_i^{k_i}, \quad r_i=\sqrt{[(x-x_i)^2+(y-y_i)^2]}.$$

$$U=\sum_{i=1}^{n} k_i\,\theta_i, \text{ where } \theta_i=\text{arc tan } \frac{y-y_i}{x-x_i}.$$

Therefore $W=V+iU=\sum_{i=1}^{n} k_i \log (z-a_i)=\log \prod_{i=1}^{n} (z-a_i)^{k_i}$.

In the case of steady streaming of water or electricity over some portion of the $x\,y$ plane the potential function is

$$V=\sum_{i=1}^{n} k_i \log r_i-\sum_{j=1}^{m} h_j \log r_j,$$

where k_i and h_j are proportional to the capacities of the n sources and m sinks respectively. Corresponding to this

$$U=\sum_{i=1}^{n} k_i\,\theta_i-\sum_{j=1}^{m} h_j\,\theta_j,$$

and the analytic function is

$$W=\log\frac{\prod_{i=1}^{n}(z-a_i)^{k_i}}{\prod_{j=1}^{m}(z-\beta_j)^{h_j}}.$$

The sources and sinks are the roots and poles of the rational analytic function

$$\frac{\prod_{i=1}^{n}(z-a_i)^{k_i}}{\prod_{j=1}^{m}(z-\beta_i)^{h_j}}.$$

Every rational analytic function can be put in this form, and since it is possible to station sources and sinks of appropriate relative capacity at the points a_i and β_j there is a one-to-one correspondence between the rational analytic functions and the potential functions for a finite number of sources and sinks of finite capacity. It is therefore possible to write down at once the analytic function which will define and characterize the streaming whenever the location and strength of the various sources and sinks in the $x\,y$ plane is known. By means of limits one may solve many other examples in logarithmic potential. The equipotential curves

$$\frac{r_1^{k_1}.r_2^{k_2}\ldots r_i^{k_i}\ldots r_n^{k_n}}{r_{n+1}^{h_1}.r_{n+2}^{h_2}\ldots r_{n+j}^{h_j}\ldots r_{n+m}^{h_m}}=C_1,$$

are the Cassinian ovals of fractional order. Their foci are the points a_i and β_j, $i=1, 2, 3, \ldots n$, $j=1, 2, \ldots m$. The stream lines form the set of hyperbolas of higher order the polar equation for which is

$$k_1\,\theta_{k_1}+k_2\,\theta_{k_2}+\ldots+k_n\,\theta_{k_n}-h_1\,\theta_{h_1}-h_2\,\theta_{h_2}-\ldots-h_m\,\theta_{h_m}=C_2.$$

Each branch of one of these hyperbolas joins a source with some sink, the point at infinity counting as a sink when the streaming is steady and the sources have a greater total capacity than the sinks in the finite part of the plane.

GENERALIZATION OF A SIMPLE FORMULA.

By ARNOLD DRESDEN, University of Wisconsin.

§ 1. The inequality

$$a^2+b^2 \geq 2ab \tag{1}$$

is well known. The object of this paper is to give a simple proof of a generalization of this formula, viz., of the formula:

$$C_{n,j} \sum_{i=1}^{n} a_i^j \geq n \sum_{k_1, k_2, \ldots, k_j=1}^{n} a_{k_1} a_{k_2} \ldots a_{k_j}, \tag{I}$$

for $j \leq n$, where all quantities a are real and positive, j and n are positive integers, and

$$C_{n,j} \equiv \frac{n!}{j!\,(n-j)!}.$$

We shall furthermore always assume:

$$k_1 < k_2 < \ldots < k_j.$$

As special cases, we have:

$$j=2,\ n=2,\ a_1^2+a_2^2 \geq 2a_1 a_2$$

$$j=2,\ n=3,\ a_1^2+a_2^2+a_3^2 \geq a_1 a_2+a_2 a_3+a_3 a_1$$

$$j=2,\ n=4,\ 6(a_1^2+a_2^2+a_3^2+a_4^2) \geq 4(a_1 a_2+a_1 a_3+a_1 a_4+a_2 a_3 + a_2 a_4+a_3 a_4)$$

$$j=3,\ n=3,\ a_1^3+a_2^3+a_3^3 \geq 3a_1 a_2 a_3.$$

§ 2. We shall first prove that (I) holds for $j=p+1$ and $n=p+1$, if it holds for $j=n=p$; *i. e.*, if we think of the positive integers arranged as double indices in a square matrix, we shall prove the formula to hold for the elements along the principal diagonal.

We have for $n=j=p$, since $C_{p,p}=1$:

$$\begin{cases} a_1^p+a_2^p+\ldots+a_p^p \geq p a_1 a_2 \ldots a_p \\ a_1^p+\ldots+a_{i-1}^p+a_{i+1}^p+\ldots+a_{p+1}^p \geq p a_1 \ldots a_{i-1}\, a_{i+1} \ldots a_p \\ a_2^p+a_3^p+\ldots+a_p^p+a_{p+1}^p \geq p a_2 a_3 \ldots a_p\, a_{p+1} \end{cases}$$

Clearly, there are $p+1$ such inequalities, one element out of the $p+1$ being omitted in each of them. Hence, if we add them, each term a_i^p ($i=1$

$\ldots p+1)$ will occur p times on the left hand side, whereas on the right-hand side, we shall have the sum of all possible products of p elements out of the $p+1$. We obtain then, after dividing by p:

$$\sum_{i=1}^{p+1} a_i^p \geq \sum_{\substack{i_1, i_2 \ldots i_p = 1 \\ i_1 < i_2 \ldots < i_p}}^{p+1} a_{i_1} a_{i_2} \ldots a_{i_p}$$

Multiplying both sides of this inequality by $\sum_{i=1}^{p+1} a_i$, we get, since all a_i's and therefore $\sum a_i$ are positive:

$$\sum_{i=1}^{p+1} a_i^{p+1} + \sum_{\substack{i,j=1 \\ i<j}}^{p+1} a_i a_j (a_i^{n-1} + a_j^{n-1}) \geq (p+1) \prod_{i=1}^{p+1} a_i + \sum_{\substack{i,j=1 \\ i<j}}^{p+1} (a_i^2 + a_j^2) \prod_{k=1}^{p+1}{}'' a_k,$$

where $\prod_{k=1}^{p+1}{}'' a_k \equiv a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots a_{j-1} a_{j+1} \ldots a_{p+1}$, *i. e.* the product of all a_i's $(i=1 \ldots p+1)$, except a_i and a_j. Now, we know from (1):

$$a_i^2 + a_j^2 \geq 2 a_i a_j.$$

Hence, we get:

$$\sum_{i=1}^{p+1} a_i^{p+1} + \sum_{\substack{i,j=1 \\ i<j}}^{p+1} a_i a_j (a_i^{n-1} + a_j^{n-1}) \geq (p+1) \prod_{i=1}^{p+1} a_i + 2C_{p+1,2} \prod_{i=1}^{p+1} a_i.$$

Furthermore, we have

$$(a_i^p - a_j^p)(a_i - a_j) \geq 0.$$

Hence, $a_i^{p+1} + a_j^{p+1} \geq a_i a_j (a_i^{p-1} + a_j^{p-1})$. Therefore, we may write:

$$\sum_{i=1}^{p+1} a_i^{p+1} + \sum_{\substack{i,j=1 \\ i<j}}^{p+1} (a_i^{p+1} a_j^{p+1}) \geq (p+1) \prod_{i=1}^{p+1} a_i + 2C_{p+1,2} \prod_{i=1}^{p+1} a_i.$$

On the left-hand side, every a_i^{p+1} $(i=1 \ldots p+1)$ occurs now $p+1$ times; remembering, therefore, that

$$2C_{p+1,2} = (p+1)p,$$

we obtain, after dividing through by $p+1$:

$$\sum_{i=1}^{p+1} a_i^{p+1} \geq (p+1) \prod_{i=1}^{p+1} a_i,$$

which is the desired result, since also $C_{p+1,\,p+1}=1$.

§ 3. Secondly, we shall prove that if (I) holds for $j=p$ and $n=m$, it will also hold for $j=p$ and $n=m+1$; *i. e.*, we shall prove the formula for the double indices above the principal diagonal of their matrix. We have now:

$$\left\{\begin{array}{l} C_{m,p}(a_1{}^p+\ \ldots\ \ldots\ \ldots\ +a_m{}^p) \geqq m \sum\limits_{k_1\ldots k_p=1}^{m} a_{k_1}\, a_{k_2} \ldots a_{k_p}, \\ C_{m,p}(a_1{}^p+\ldots+a_{i-1}{}^p+a_{i+1}{}^p+\ldots a_{m+1}{}^p) \geqq m \sum\limits_{k_1\ldots k_p=1}^{m+1\,\prime} a_{k_1}\, a_{k_2} \ldots a_{k_p}, \\ C_{m,p}(a_2{}^p+\ \ldots\ \ldots\ \ldots\ +a_{m+1}{}^p) \geqq m \sum\limits_{k_1\ldots k_p=2}^{m+1} a_{k_1}\, a_{k_2} \ldots a_{k_p}, \end{array}\right.$$

where $\sum\limits_{k_1\ldots k_p=1}^{m-1\,\prime}$ indicates that $k_1, \ldots, k_p$ are to take all values from 1 to $m+1$, *except i.*

We have this time $m+1$ inequalities, each one of the elements $a_1, \ldots, a_{m+1}$ being omitted in one of the inequalities. Hence, if we add them, each term $a_i{}^p$ $(i=1, \ldots, m+1)$ will occur m times on the left-hand side. On the right-hand side of each inequality, there are $mC_{m,p}$ terms; in the sum there willbe $(m+1)mC_{m,p}$ terms, each of which is a product of p elements out of the $m+1$. Of those, there are only $C_{m+1,\,p}$ different ones. Since there is symmetry with respect to every element, each of these products is repeated $\dfrac{(m+1)mC_{m,p}}{C_{m+1,\,p}}$ times. Hence, we obtain:

$$mC_{m,p} \sum_{i=1}^{m+1} a_i{}^p \geqq \frac{(m+1)mC_{m,p}}{C_{m+1,\,p}} \prod_{i=1}^{m+1} a_i{}^p \text{ or } C_{m+1,\,p} \sum_{i=1}^{m+1} a_i{}^p \geqq (m+1) \prod_{i=1}^{m+1} a_i{}^p,$$

which was to be proved.

The results of § 2 and § 3 establish the formula, which it was desired to prove.

§ 4. In part II of Chrystal's *Algebra* (second edition, 1906), we find on page 46:

$$\frac{a_1+a_2+\ldots+a_k}{k} \geqq (a_1\, a_2 \ldots a_k)^{1/k},$$

and on page 49:

$$\frac{a_1{}^m+a_2{}^m+\ldots+a_k{}^m}{k} \geqq \left(\frac{a_1+a_2+\ldots+a_k}{k}\right)^m.$$

Combining these two formulae for $k=m$, we obtain (I) for $j=n$. The general formula (for $n \geqq j$) is not in the literature so far as I know. It might still be desired to establish a formula which shall hold also for $n<j$.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

340. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Let $S_{n-1}=1^{n-1}+2^{n-1}+3^{n-1}+\ldots+(n-1)^{n-1}$. Find n if $S_{n-1}-(n-1)$ is a multiple of n^2.

No complete solution of this problem has been received.

341. Proposed by O. L. CALLICOTT, Gettysburg, South Dakota.

Prove that the sum of the series, $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\ldots$ to infinity = the sum of the series $\frac{1}{2}.1+\frac{1}{2^2}.\frac{1}{2}+\frac{1}{2^3}.\frac{1}{3}+\frac{1}{2^4}.\frac{1}{4}+\ldots$ to infinity.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

$$\frac{1}{(n+1)\ (n+2)}\equiv\frac{1}{n+1}-\frac{1}{n+2}.$$

Hence, by substituting for n, 0, 1, 2, 3, ..., and adding, we have

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots.=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots.=\log 2.$$

Now $-\log(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots$, which for $x=\frac{1}{2}$, becomes

$$-\log(\tfrac{1}{2})=\log 2=\tfrac{1}{2}+\tfrac{1}{2}(\tfrac{1}{2})^2+\tfrac{1}{3}(\tfrac{1}{2})^3+\ldots$$

This proves the proposition.

Also solved by V. M. Spunar.

GEOMETRY.

367. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

The tangents from a point A to a circle are bisected by a line XYZ, which cuts a chord in X and the tangents at its extremities in Y, Z. Show that $XAY=XAZ$, or $XAY=\pi-XAZ$. Also, reciprocate with respect to A.

Solution by S. LEFSEHETZ, Fellow in Clark University.

Let B, D be the points where the chord and circle meet. From Z draw ZC parallel to YD. The $\angle CBZ=\angle CDY=\angle BCZ$.

$\therefore CZ=BZ$. Also $ZB=ZA$, and $YB=YA$, hence is radical axis of circle and point A.

$\therefore \frac{XY}{XZ}=\frac{YD}{ZC}=\frac{YD}{ZB}=\frac{YA}{ZA}$, which shows that AX is one of the bisectors of the angle YAZ. The other is AX', X' being the point where the polar of X relative to the circle cuts XYZ.

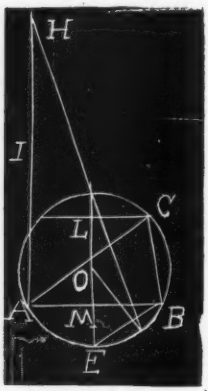
Reciprocally, given a line XZ which does not cut a circle, O, the points of this line which are conjugate relatively to O form on YZ a division in involution. There is a point A from which segments like XX' are seen under a right angle, and YZ is the radical axis of this point and the circle.

368. Proposed by G. I. HOPKINS, Professor of Mathematics and Astronomy, Manchester High School, Manchester, N. H.

It is required to construct the triangle having given, base, vertical angle, and ratio of its altitude to sum of the other two sides.

Solution by the late G. B. M. ZERR, Ph.

Let $AB=a$ be a given base; C, the given vertical angle; x, y, z the two other sides and altitude, respectively; m the given ratio so that $x+y=mz$.



Now $xy\sin C=az$, $a^2=x^2+y^2-2xy\cos C$, $x^2+2xy+y^2=m^2z^2$. Hence $a^2=m^2z^2-2az\cot\frac{1}{2}C$.

$$\therefore z=\frac{a}{m}(\cot\tfrac{1}{2}C+\operatorname{cosec}\tfrac{1}{2}C)=\frac{a}{m}\cot\tfrac{1}{4}C.$$

Construction: On AB construct a segment containing the angle C. Draw the diameter DE perpendicular to AB. Bisect arc EB with radius OF at point F. Draw AI perpendicular to AB, and draw FD to meet AI produced in H. Then $\angle AHF=\angle EDF=\frac{1}{4}C$, and $AH=a\cot\frac{1}{4}C$. Find z a fourth proportional to M, AH and unity.

Take $ML=z$, and through L draw LC perpendicular to ML, meeting the circle in C. Join CA, CB, giving the required triangle ABC.

Also solved by A. H. Holmes.

CALCULUS.

295. Proposed by C. E. FLANAGAN, Wheeling, W. Va.

A hawk can fly v feet per second, a hare can run v' feet per second. The hawk, when a feet vertically above the hare, gives chase and catches the hare when the hare has run b feet. Find the length of the curve of pursuit. [Echols' *Differential and Integral Calculus*, page 253, Ex. 20.]

I. Solution by FRANK LOXLEY GRIFFIN, Ph. D., Williams College.

The time that the hare runs is clearly b/v' seconds, which is also the time of the hawk's flight; thus, S, the length of the curve of pursuit, is vb/v' feet.

Remark. For the given area a to be consistent with v, v' and b there is a certain necessary condition (found below), which if fulfilled renders part of the data superfluous, as shown by the preceding solution. Further, if b be supposed unknown, we can still find S in terms of a, v, v'.

Let the hare's path be taken as the X-axis starting from $(0, 0)$ so that the hawk starts from $(0, a)$; and let their coordinates be, respectively, $(X, 0)$ and (x, y) after t seconds. Then the hawk has flown a distance s:

$$X=x-y\frac{dx}{dy}=v's/v\ldots(\text{I}).$$

Differentiating (I) with respect to y gives: $\frac{v'}{v}\frac{ds}{dy}=-y\frac{d^2x}{dy^2}$; in which put $\frac{dx}{dy}=p$ and $\frac{v'}{v}=k<1$, noting that $\frac{ds}{dy}=-\sqrt{(1+p^2)}$, and separate variables: $kdy/y=dp/\sqrt{(1+p^2)}$. Integrating:

$$\log[p+\sqrt{(1+p^2)}]=\log(cy)^k,\ \text{or}\ p+\sqrt{(1+p^2)}=(cy)^k\ldots(\text{II}).$$

In (II) invert both members and rationalize denominators: $\sqrt{(1+p^2)}-p=(cy)^{-k}$, whence $2p=(cy)^k-(cy)^{-k}$.

Again separating and integrating we have

$$2c(x+d)=\frac{(cy)^{1+k}}{1+k}-\frac{(cy)^{1-k}}{1-k}\ldots(\text{III}).$$

The constants c, d are determined by the fact that, when $x=0$, $y=a$ and $p=0$. Thus (II) gives $c=1/a$, and (III) gives $d=-ak/(1-k^2)$. We find b from the fact that $(b, 0)$ must satisfy (III): $b=-d=ak/(1-k^2.)$ [This is the above-mentioned necessary condition.]

Since $S=vb/v'=b/k$, we have $S=a/(1-k^2)$ or $v'^2a/v^2-v'^2)$.

Also solved by G. B. M. Zerr, E. L. Sherwood, and V. M. Spunar.

II. Solution by C. E. FLANAGAN.

This problem as above stated is not a Calculus problem at all, but one of simple proportion; for, plainly, if s=the length of the hawk's path, we have, $s:b=v:v'$. Hence $s=b\frac{v}{v'}$. When b is *unknown* and only a, v, and v' are given, then we have a Calculus problem to solve, and under these conditions I offer the following

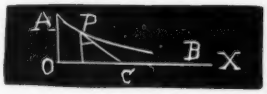
Solution. Let s=length of hawk's path and $\frac{v'}{v}=n$. From similar triangles we have,

$$-dy:y=dx:ns-x.$$

$$\therefore ns-x=-y\frac{dx}{dy};\ \therefore nds-dx=-yd\left(\frac{dx}{dy}\right)-\frac{dx}{dy};\ \text{or since } \frac{dx}{dy}dy=dx,\ nds$$

$$=-yd\left(\frac{dx}{dy}\right).\quad \text{But } ds=\sqrt{1+\frac{dx^2}{dy^2}}dy.\quad \therefore n\sqrt{1+\frac{dx^2}{dy^2}}dy=-yd\left(\frac{dx}{dy}\right);$$

$$\therefore -n\frac{dy}{y}=\frac{d\left(\frac{dx}{dy}\right)}{\sqrt{1+\frac{dx^2}{dy^2}}}.$$



Integrating, $c-n\log y=\log\left(\frac{dx}{dy}+\sqrt{1+\frac{dx^2}{dy^2}}\right)$. Now when $y=a$, $\frac{dx}{dy}=0$; $\therefore c=n\log a$.

$$\therefore n\log a-n\log y=\log\left(\frac{dx}{dy}+\sqrt{1+\frac{dx^2}{dy^2}}\right);$$

$$\log\left(\frac{a}{y}\right)^n=\log\left(\frac{dx}{dy}+\sqrt{1+\frac{dx^2}{dy^2}}\right);$$

$$\frac{a^n}{y^n}=\frac{dx}{dy}+\sqrt{1+\frac{dx^2}{dy^2}};\ \therefore \sqrt{1+\frac{dx^2}{dy^2}}=\frac{a^n}{y^n}-\frac{dx}{dy};$$

$$1+\frac{dx^2}{dy^2}=\frac{a^{2n}}{y^{2n}}-2\frac{a^n}{y_n}\frac{dx}{dy}+\frac{dx^2}{dy^2};\quad 2\frac{dx}{dy}=\frac{a^n}{y^n}-\frac{y^n}{a^n};$$

$2dx=a^ny^{-n}dy-a^{-n}y^ndy$. Hence, integrating, we get

$$2x+c'=\frac{a^n}{1-n}y^{1-n}-\frac{a^{-n}}{1+n}y^{1+n}.$$

Now when $x=0$, $y=a$.

$$\therefore c'=\frac{a}{1-n}-\frac{a}{1+n}=\frac{2an}{1-n^2};$$

$\therefore 2x+\frac{2an}{1-n^2}=\frac{a^n}{1-n}y^{1-n}-\frac{a^{-n}}{1+n}y^{1+n}$, which is the equation of the curve of pursuit. From this equation, when $x=ns$, $y=0$, we get $2ns+\frac{2an}{1-n^2}=0$; $\therefore s=-\frac{a}{1-n^2}$ =length of the hawk's path; and $b=ns=-\frac{an}{1-n^2}$ =length of the hare's path. The minus sign before the fraction shows the distances are measured from B toward A and O.

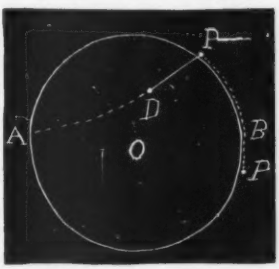
To get the formula given by Mr. Echols, we have

$$n^2=\frac{b^2}{s^2}=1+\frac{a}{s}. \quad \therefore s^2-as=b^2; \quad \therefore s=\pm\sqrt{\frac{a^2}{4}+b^2}+\frac{a}{2}.$$

NOTE.—We have published these two solutions for the purpose of comparison. Mr. Flanagan contends that the problem does not belong to the Calculus, which in a sense is right. Yet had b been the unknown quantity, the problem would certainly have been one involving Calculus. The fact that b is given, imposes a restriction on the given quantities.

Of the problem, "A dog and duck are at the opposite ends of the diameter of a circular pond. How far must the dog swim to catch the duck, the dog swimming n times as fast as the duck and directly towards it at all times, $n>1$," Mr. Flanagan offers the following solution, which we publish for criticism:

Let the diameter AB be an inextensible string and P and P' two moving pencils. The pencils move in opposite directions around the circle. They are together at B at the start. They move, say, at equal speeds. The pencil P' is fastened to the end of the string, but the string moves freely against the end of the pencil P. The heavy pellet A is pulled to the position D along the dotted line whose length at any moment is equal to S, the line DP being tangent to the path of the pellet. It is plain that $PP'=S$ and that $DP=2r-s$. Now when the pellet has reached the circumference $DP=2r-s=0$. Therefore, $s=2r$ when the pursued travels half as fast as the pursuer. In like manner it can be shown that the length of the path of the pursuer is always equal to the diameter of the circle when the pursuer travels as fast or faster than the pursued.



MECHANICS.

247. Proposed by C. N. SCHMALL, New York City.

A cylinder of height h and radius r is standing on a horizontal seat in a railway car while the train is getting under way with an acceleration f. Show that the cylinder will not be disturbed if

$$f<\mu g, \text{ and } f<2rg/h,$$

where μ is the coefficient of friction.

I. Solution by S. G. BARTON, Ph. D., Clarkson School of Technology.

Let m be the mass of the cylinder; then the forces acting are mf horizontally at center of gravity $\frac{1}{2}h$ above base, friction $mg\mu$, horizontally in opposite direction at the base, and mg vertically downward.

The cylinder will not upset unless the moment of the accelerating force about the point of base in the direction of the motion exceeds the moment of mg, that is unless

$$mf\frac{h}{2}>mgr, \text{ that is, it will not upset if } f<\frac{2rg}{h}.$$

It will not slide unless the accelerating force is greater than friction, that is, $mf>mg\mu$; or it will remain at rest if $f<g\mu$.

II. Solution by the late G. B. M. ZERR, Ph. D.

Let F=force that causes sliding or toppling. Then $F=fm$ where mg =weight of cylinder. For sliding to begin, $F=mg\mu$.

$\therefore fm=mg\mu$ or $f=\mu g$. Hence, if $f<\mu g$ there is no sliding.

For toppling to just start, $\frac{1}{2}F.h=rmg$.

Hence, $\frac{1}{2}mfh=rmg$, and $f=2rg/h$.

Then if $f<2rg/h$, there will be no toppling.

No solution of No. 248 has yet been received.

PROBLEMS FOR SOLUTION.

ALGEBRA.

345. Proposed by E. B. ESCOTT, Professor of Mathematics, University of Michigan.

Solve the simultaneous equations:

$$x^3+y^2z+zw^2-xz^2-2wxy=a\ldots(1).$$
$$y^3+z^2w+wx^2-yw^2-2xyz=b\ldots(2).$$
$$z^3+w^2x+xy^2-zx^2-2yzw=c\ldots(3).$$
$$w^3+x^2y+yz^2-wy^2-2zwx=d\ldots(4).$$

346. Proposed by E. B. ESCOTT, Professor of Mathematics, University of Michigan.

Solve completely by quadratics:

$$\frac{2x}{1-x^2}=y;\quad \frac{2y}{1-y^2}=z;\quad \frac{2z}{1-z^2}=w;\quad \frac{2w}{1-w^2}=x.$$

347. Proposed by GUSTAVE JACOBSON, A. M., Public Accountant, Chicago, Ill.

A corporation needing some additional capital for a short term of years, issues $300,000 of debenture bonds carrying 6% interest, and payable 1/5 each year for 5 years. Coupons are attached to the bonds maturing every six months; the bonds are sold at 90 flat. What average rate of interest does the company pay for the money, including interest on interest?

348. Proposed by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.

Prove the following relation between Jacobi's symbols:

$$\left(\frac{d}{n}\right)=(-1)^{m(d-1)/2}\left(\frac{d}{2md-n}\right)$$

where d and n are positive odd numbers and $2md>n$.

GEOMETRY.

377. Proposed by S. A. COREY, Hiteman, Iowa.

Let AB, BC, CD, DE, EA be the sides of a pentagon, plane or gauche. From A draw AF, AG, AH, parallel to, and of the same length and currency as BC, CD, DE, respectively. Bisect AE at K. Draw KB, KF, KG, and KH. Prove that $KB^2+KF^2+KG^2+KH^2=AB^2+BC^2+CD^2+DE^2$.

378. Proposed by G. I. HOPKINS, A. M., Instructor in Mathematics and Astronomy, Manchester High School Manchester, N. H.

In the triangle AED, the lines BE and CE are drawn to the points B and C in the base of the triangle. If $AE=100$, $ED=125$, $BC=60$, and $\angle AEC=\angle BED=$a right angle, compute AB, BE, EC, and CD.

CALCULUS.

302. Proposed by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.

Find an algebraic integral of

$$LCy''+[18L(s-x^2)-2kC]y'+[L(\tfrac{2}{3}k-10x)-8k(s-x^2)]y=0,$$

where $L=2kx+3s-k^2$, $C=12sx-4x^3-t$, $k^4-6sk^2-tk-3s^2=0$. The roots of the determining equation are 0, $-\frac{1}{2}$ for a root of $C=0$; 0, 2 for a root of $L=0$; and 1 and $\frac{3}{2}$ at infinity.

303. Proposed by C. N. SCHMALL, New York City.

If $A=\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$, express A in terms of Gamma-functions. [Part of ex. 39, p. 474, Bromwich *Theory of Infinite Series.*]

MECHANICS.

254. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

On a smooth horizontal plane are touching each other two balls, equal, uniform, inelastic, spherical, radius x, mass M. A third uniform, smooth, elastic ball, radius y, mass m, is placed with its centre vertically above the point of contact of those balls. Its radius may vary as long as its mass remains m. When is the velocity of each inelastic ball after impact a maximum?

355. Proposed by the late G. B. M. ZERR, Ph. D.

Find the current given by a battery $\frac{1}{2}n(n+1)$ cells arranged in n rows as follows: First row, 1 cell; second row, 2 cells; third row, 3 cells; nth row, n cells. The electromotive force of each cell is E and its resistance r, while the resistances of the wires joining the positive and negative ends of the rows to the poles are as follows: First row, nr; second row, $(n-1)r$; third row, $(n-2)r$, ..., $(n-2)$nd row, $3r$; $(n-1)$st row, $2r$; nth row, r. The resistance of the two wires joining the poles to complete the circuit is r for each.

NOTES AND NEWS.

Dr. A. M. Hiltebeitel (Princeton) has been appointed an instructor in mathematics at the University of Pennsylvania.

The Chicago Section of the American Mathematical Society will hold its meeting at Minneapolis on Wednesday, Thursday, and Friday, December 28th to 30th, 1910. On Wednesday there will be a joint meeting with Section A of the American Association to hear the vice-presidential address of Professor E. W. Brown. On Friday afternoon there will be a joint meeting with Sections A and D to hear the report of the committee on Teaching of Mathematics to Engineering Students, which was appointed two years ago at Chicago. Professor E. V. Huntington, of Harvard University, is chairman of this committee. S.

At the Mathematical Conference, recently held at the University of Chicago, there were reports from several members of committees working under the International Commission on the Teaching of Mathematics. Professor J. W. A. Young, of the University of Chicago, who is one of the American Commissioners, gave an interesting account of the work which has been accomplished to date in other countries and a statement of the general scheme which has been employed in this country. Other speakers for various committees were: Charles Ammerman, McKinley High School, St. Louis, Mo.; Walter W. Hart, University of Wisconsin; Herbert E. Cobb, Lewis Institute, Chicago; John Schobinger, Harvard School, Chicago; Ernest R. Breslich, University High School, Chicago; Hiram B. Loomis, Hyde

Park High School, Chicago; Emma M. Cowles, Milwaukee-Downer College. These preliminary reports have for the most part been completed and will be presented by the American Commissioners before the International Commission at a meeting to be held some time next spring. The final report is to be made at the International Congress of Mathematicians at Cambridge, England, in the summer of 1912. S.

At the meeting of the Central Association of Science and Mathematics Teachers at Cleveland, Ohio, on November 25-26, Professor David Eugene Smith, of Teachers College, Columbia University, gave an address before the Mathematics Section on the Fundamental Reason for Teaching Mathematics in the Secondary Schools. There was also presented two important committee reports, the first on the results thus far obtained in connection with improvements in the teaching of mathematics along the lines by previous investigations, and second, on the standarizing of mathematical notation in elementary subjects. In connection with this meeting there was a session of the committee of fifteen on Geometry Syllabus working under the auspices of the N. E. A.

At the Northern Illinois Teachers Association meeting held at Chicago on November 4th and 5th, the topic for discussion throughout all the sessions was, "The Place of the Concrete in Education." Much stress was laid upon the fact that in every branch of the curriculum it is desirable to base the educational processes on concrete situations so far as is possible. It is interesting to know that this proposition has been taken up by certain teachers in the department of political economy and has been applied most effectually in reducing the teaching of this abstract subject to a concrete basis.

In the issue of *Science* for August 19, 1910, there is an interesting report on the granting of Doctor's degree by American Universities. There are numerous tables and interesting deductions drawn from the figures given. The following items are gathered from these details: The total number of doctorates in the last 13 years is 3833, of which 1787 are in the sciences, including mathematics. Of this total number Chicago has conferred 490, Columbia 480, Harvard 453, Yale 421. Of the science degrees Chicago has conferred 243, Johns Hopkins 220, Columbia 189, Yale 179, Harvard 178. In mathematics there were altogether 23 doctorates conferred by all the Universities in the year 1908, 21 in 1909, 20 in 1910, with an average of 12 for the past 10 years.

In the issues of *Science* for November 4th and 11th, 1910, there is an interesting discussion relative to the revision of the biographical index of the American Men of Science. There are numerous details showing the distribution of the first thousand and the second thousand leading men of science, according to their occupations, their geographical locations, and the

institutions with which they are connected. The states containing 50 or more of the first thousand leading men of science are in order, New York 183, Massachusetts 165, Maryland (including the District of Columbia) 148, Illinois 77, Pennsylvania 60, California 50, Connecticut 50. The institutions containing 30 or more men of the first thousand are in order, Harvard 79, Columbia 48, Chicago 47, Yale 38, Cornell 35, Johns Hopkins 33, Wisconsin 30. Of the Universities having the strongest departments according to this basis, Harvard has five, standing first, namely, physics, botany, zoology, physiography and pathology. Chicago has two standing first, namely, mathematics and astronomy. Columbia, Johns Hopkins and Massachusetts Institute of Technology each have one department standing first, namely. psychology, anatomy, and chemistry respectively. The six leading universities ranked according to their total scientific strength on this basis are in order, Harvard, Chicago, Columbia, Johns Hopkins, Yale, and Cornell. S.

The opening session of the Minneapolis Meeting of the American Association for the Advancement of Science will be held on Tuesday evening, December 27. The section for Mathematics and Astronomy will begin its sessions the following morning at 10 a. m. The program for all the meetings of the Association is expected to be ready on the opening day. Papers which are to be read before Section A, should reach the Secretary, Professor G. A. Miller, before December 5.

In a recent circular issued by B. G. Teubner, of Leipzig, Germany, it is announced that their supply of the first volume of the German edition of the great mathematical encyclopedia is exhausted and they advise that those interested should secure the corresponding part of the French edition. Thirty-five volumes of the latter are in preparation. They are published in parts of about 160 pages. Fifteen of these relating mainly to arithmetic and algebra have already appeared and seventeen additional parts are said to be now in press.

BOOKS AND PERIODICALS.

Text Book of Physics. By C. E. Linebarger, Lake View High School, Chicago. 8vo. Cloth, viii+471 pages. Price, $1.50. Boston: D. C. Heath & Co.

In writing an elementary text-book on physics, one is at a loss to know just what to select from the great abundance of important material at one's disposal. In the present text, the author has succeeded very well in selecting from the great wealth of material that which is interesting and important, and his treatment of the subject aims to be "intensive rather than extensive." The definitions are clearly stated, and the principles and laws have been logically established. The illustrations are numerous and true to reality. At the close of each chapter is a list of questions and problems designed to test the student's knowledge of the principles. F.

The Calculus for Beginners. By J. W. Mercer, M. A., Head of the Mathematical Department, Royal Naval Academy, Dartmouth. 8vo, cloth. xiv+440 pp. Price, $2.00. Cambridge, England: The University Press. G. P. Putnam's Sons, New York, American Agents.

The author of this book believes that it is more important for the beginner to understand clearly what the processes of the calculus mean and what the calculus can do for him than a facility in manipulation of the processes and a wide acquaintance with them. So he has written this book with the aim to emphasize the underlying principle of the calculus and impress this principle on the mind of the beginner. He has, therefore, given many pages to the ideas of rates of change and of the limiting value of the ratio of two continually diminishing quantities, and in the various illustrations given it is hoped that the beginner will emerge from darkness into a clear insight of the subject.

The book is intended primarily for those students who are likely to take up engineering, and no great amount of previous mathematical knowledge is assumed.

Many of the illustrations are arithmetical or geometrical and many are drawn from the knowledge at the command of most boys at the age at which they are expected to take up the study of the calculus. F.

The Elements of the Theory of Algebraic Numbers. By Legh Wilber Reid, Professor of Mathematics in Haverford College, with an Introduction by David Hilbert, Professor of Mathematics in the University of Göttingen. Large 8vo, cloth. xv+454 pages. Price, $3.50 net.

From the standpoint of pure mathematics, this is among the most important mathematical works that has thus far been published in America. For, until the publication of this work, the ideas developed could only be read in the German or the French. Very little attention has thus far been given to the study of the theory of Algebraic Numbers. True, courses have been given on the subject in some of our larger Universities, but no productive work of much consequence has thus far resulted. The appearance of this work will, no doubt, stimulate an interest in one of the most abstruse, and at the same time one of the most interesting subjects of pure mathematics.

The author has aimed to lead a reader, entirely unacquainted with the subject, by easy steps, to an appreciation of some of the fundamental conceptions in the general theory. In this respect, we believe that he has succeeded quite well. By a study of this book one may take up and read with better understanding the works on the general theory by such mathematicians as Hilbert, Dedekind, Kronecker, Kummer, and others.

In Hilbert's Preface to his *Die Theorie der Algebraischen Zahlenkorper* is given a brief historic survey of the subject from the time of Gauss down to the closing days of the nineteenth century.

The author of the present work deserves the thanks of the younger mathematicians who,—in view of the *new education* which is now so enthusiastically proclaimed by pedagogic theorists, to the end that no subjects should be required for graduation,—have neglected to study French and German. Not only these and those who shall follow them, but the mature mathematician as well, will welcome this book written by one who has devoted much of his time to a thorough study of a profound subject. F.

Elementary Analysis. By Percy F. Smith, Ph. D., Professor of Mathematics in the Sheffield Scientific School of Yale University, and William Antony Granville, Ph. D., Instructor in Mathematics in the Sheffield Scientific School of Yale University. 12mo, cloth. Illustrated. x+223 pp. Price, $1.50. Boston and Chicago: Ginn & Co.

This book presents a course of seventy lessons in mathematics beyond trigonometry, or in Analytical Geometry and the Calculus.

The first chapter deals with formulas of reference from Algebra and Trigonometry, the second with Cartesian coordinates; the third, with Curve and Equation, and so on subjects are taken up in Analytical Geometry until chapter seven is reached, in which differentiation is taken up. The remainder of the book is devoted to processes in the Calculus.

The authors have endeavored to give in this book as much mathematics as is needed by the average student of the natural and experimental sciences, and no student ought to be graduated from a scientific course of study who is unfamiliar with the mathematics of the text before us. F.

A Class-Book of Trigonometry. By Charles Davison, Sc. D., Mathematical Master at King Edward's High School, Birmingham. 8vo, cloth. viii+200 pp. Price, $1.00. Cambridge, Eng.: The University Press. G. P. Putnam's Sons, New York, American Agents.

The author's aim in preparing this book for public favor has been to reach the solution of the triangle as soon as possible. To this end many details usually found in works on the subject are here omitted. The book is neatly printed and bound, this being in conformity with all work done by the University Press. F.

The Fundamental Theorems of the Differential Calculus. By W. H. Young, Sc. D., F. R. S. Formerly Fellow of Peterhouse, Cambridge, Lecturer on Higher Analysis at the University of Liverpool. 8vo, paper cover. ix+72 pages. Price, 75 Cents. Cambridge, Eng.: Cambridge University Press. G. P. Putnam's Sons, New York, American Agents.

This is No. 11 of the Cambridge Tracts in Mathematics and Mathematical Physics, under the general editorship of J. G. Leahman and E. T. Whittaker.

The author, in this tract, aims rather at rigidity of proof and novelty of treatment than simplicity of presentation. To carry out complete rigor in some of the demonstrations some of the Theory of Sets of Points are introduced.

Teachers of the Calculus who desire a rigorous presentation of the Fundamental Theorems of the Calculus will find this tract very helpful and suggestive. F.

Orders of Infinity. 'The Infinitärcalcül' of Du Bois-Reymond. By G. H. Hardy, M. A., F. R. S. Fellow and Lecturer of Trinity College, Cambridge. 8vo, paper cover. ii+62 pages. Price, 75 Cents. Cambridge: The Cambridge University Press. G. P. Putnam's Sons, American Agents.

This is No. 12 of the Cambridge Tracts referred to above, and has for its object the presentation and preservation of some of the most important discoveries of the great mathematician, Du Bois-Reymond. The author has attempted to bring the Infinitärcalcül up to date, giving explicit statements and careful proofs of some of the general theorems, "the truth of which Du Bois-Reymond seems to have tacitly assumed." F.

Practical Measurements. By A. W. Siddons, M. A., Mathematical Master at Harrow School, Late Fellow of Jesus College, Cambridge, and A. Vassals, M. A., Senior Science Master at Harrow School. 8vo, board cover. xv+60 pages. Price, 45 Cents. Cambridge, Eng.: The University Press. G. P. Putnam's Sons, New York, American Agents.

This book is an attempt, we are told, to arrange a course of weighing and measuring to further the closer relationship between mathematical and science teaching. To

this end a course of experiments has been so arranged that the boy will gain a clearer idea about many points which were somewhat obscure when met with in the mathematical lessons. When time can be secured the course will certainly prove to be of great value to students. F.

Eléments de Calcul Vectoriel, Avec de Nombreuses Applications, A La Géométrie, A La Mécanique, et A La Physique Mathématique. Par C. Burali-Forti, Professeur a L'Acadèmie Militaire de Turin et R. Marcolongo, Professeur de Mecanique Rationnelle a L'Universitè de Naples. Traduit de L'Italien par S. Lattès Maitre de Conferences a L'Universitè de Montpellier. 8vo, paper cover. vii+229 pages. Prix, 8f=$1.60. Paris, France: A. Hermann et Fils.

This work develops the principles of Vector Analysis and applies them in the second part of book to the solution of problems in Differential Geometry, Mechanics and Mathematical Physics. F.

The Imaginary in Geometry. By Ellery W. Davis, Lincoln, Nebraska. Paper cover. 58 pages. Reprinted from the University Studies, Lincoln, Nebraska. Vol. x, No. 1. January, 1910.

A paper of much interest and value. F.

A Short Table of Integrals. By B. O. Pierce, Hollis Professor of Mathematics and Natural Philosophy in Harvard University. Second Revised Edition. 8vo, cloth. 144 pages. Price, $1.50. Boston and Chicago: Ginn & Co.

This excellent little volume has been made even more useful by this revised edition, in which the author has added a few more formulas, e. g., pages 15, 30, 61, 110, and 111, and also a table of natural logarithms of all numbers of three digits between 1 and 5, and of all numbers of two digits between 5 and 10. F.

Anharmonic Coördinates. By Lieut.-Colonel Henry W. L. Hime, (Late) Royal Artillery, (author of Outlines of Quaterions, etc.). 8vo, cloth. xiv+127 pages. Price, $3.00. New York and London: Longmans, Green & Co.

Since no work had been written on the subject of which this work treats, and since only brief references are given by several authors on Quaternions, and none at all by writers on ordinary coördinate Geometry, the author felt that an invention by so great a mathematician as Sir W. R. Hamilton should not be allowed to be forgotten. The method is applied to the well-known geometrical theorems of conics including their special forms. The book is well written and the treatment quite lucid. F.

The Former Rates of the Earth's Rotation and Their Bearings on its Deformation, and the Bearings of Molecular Activity on Spontaneous Fission in Gaseous Spheroids. By Thomas Chrowder Chamberlain, Professor of Geology, University of Chicago. Extraction from Publication 107 of the Carnegie Institution of Washington. 8vo, paper cover. 59 pages+5 pages.

These two articles, which are very important contributions to cosmogony, constituted a part of Professor Chamberlain's argument in support of his planetesimal theory of the solar system, a theory which seems to be quite generally admitted to supercede the Laplacian, or Nebular Hypothesis. In the first paper, the author arrives at the conclusion,

(a) the frictional value of the water-tides have only a negligible effect on the earth's rotation; (b) that the tides of lithosphere are chiefly elastic strains and have little retardative value, while the tides of the atmosphere are too small to be measured; and (c), that geological evidences indicate that there has been no such change in the rate of the earth's rotation during its known history as to require it to be seriously considered in the study of the earth's deformations. F.

Elements of Plane and Spherical Trigonometry. By David A. Rothrock, Ph. D., Professor of Mathematics, Indiana University, Bloomington, Indiana. 8vo, cloth. xi+147 pages +99 pages of Tables. New York: The Macmillan Co.

The author has endeavored to prepare a text, which would serve as a basis for fifty or sixty-hour courses. Emphasis is placed upon drill work in trigonometric identities, and attention is given to trigonometric applications to practical problems, and approximate calculations. Chapter IX discusses briefly the theory of analytical trigonometry. F.

ERRATA.

August-September number, p. 170, line 14, should read:

$$\frac{1}{a+\frac{1}{a}}+\frac{1}{\left(a+\frac{1}{a}\right)\left(a^2+\frac{1}{a^2}\right)}+,..=\left(a-\frac{1}{a}\right)\left[\ ...\ \right]$$

P. 171, line 8, denominator of last term of series should be

$$\frac{1}{16.254.64514}$$

i. e. the third factor 645 should be omitted.

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MATHEMATICS IN PORTUGAL.

By G. A. MILLER, University of Illinois.

The fact that Portugal has so recently joined the list of republics increases interest at this time in the intellectual development of this little country with such a turbulent history. Her maritime eminence several centuries ago implies the use of mathematics at an early date and awakens the hope that we may find here independent developments of unusual value. A people that before the close of the fifteenth century found a route to India by sailing around the southern part of Africa must have possessed at an early date the basic elements of mathematics and astronomy. While the mathematical history of Portugal does not contain names that can be classed with Euler, Lagrange, Gauss, or Cauchy, yet it presents some names with which the mathematician is familiar. As such a name we may mention that of Nonius, who discovered before the middle of the sixteenth century, the fundamental properties of the important curves called loxodromes, which are important in navigation.*

Under the title "Les Mathématique en Portugal" M. Rodolphe Guimareas published in 1909 a second edition of a work which was originally prepared for the Universal Exposition of Paris in 1900. Nearly a hundred pages of this second edition are devoted to a sketch of the development of mathematics in Portugal. This is followed by a list of books and articles published by Portuguese writers together with occasional brief sketches as regards content or history. This list covers about 500 pages and constitutes a very useful aid towards obtaining a knowledge of the mathematical advances in this country.

From this list and from the historical sketch which precedes it, one can readily see that the Portuguese have been more successful in applying the results discovered elsewhere than in making important advances in pure mathematics. The professors in their foremost educational institution, the University of Coimbra, founded at Lisbon in 1290, and finally transferred to Coimbra in 1557, after having been moved to and fro several times, gained

*Cf. Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. 2 (1900), p. 390.